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Coderivative Analysis of Variational Systems

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Abstract. The paper mostly concerns applications of the generalized differentiation theory in variational analysis to Lipschitzian stability and metric regularity of variational systems in infinite-dimensional spaces. The main tools of our analysis involve coderivatives of set-valued mappings that turn out to be proper extensions of the adjoint derivative operator to nonsmooth and set-valued mappings. The involved coderivatives allow us to give complete dual characterizations of certain fundamental properties in variational analysis and optimization related to Lipschitzian stability and metric regularity. Based on these characterizations and extended coderivative calculus, we obtain efficient conditions for Lipschitzian stability of variational systems governed by parametric generalized equations and their specifications.

Key words: Coderivatives, Generalized differentiation, infinite-dimensional spaces, Lipschitzian stability and metric regularity, Variational analysis

1. Introduction

This paper is devoted to some fundamental issues in variational analysis concerning stability and metric regularity of optimization-related problems. Variational analysis has been recognized as a fruitful area in mathematics mostly oriented on applications to constrained optimization and also applying optimization, perturbations, and approximation ideas to the analysis of a broad range of problems that may not be of a variational nature. We refer the reader to the book of [27], which contains a systematic exposition and thorough developments of the key feature of variational analysis in finite-dimensional spaces.

The main emphasis of this paper is variational analysis in infinite dimensions and its applications to Lipschitzian stability of variational systems governed by parametric generalized equations and variational inequalities. We are going to employ the generalized differentiation theory for nonsmooth and set-valued mappings, which is one of the most important parts of variational analysis. The main tools of our study involve *coderivatives* of set-valued mappings (multifunctions) that give adequate extensions of the classical adjoint derivative operator, enjoy a comprehensive calculus, and play a crucial role in characterizations of Lipschitzian behavior, metric regularity, and covering/openness properties of general multifunctions; see [14] and the references therein. Applications of coderivative analysis to various problems related to Lipschitzian stability and metric regularity of variational systems in finite dimensions are given in [4, 5, 9, 10, 12, 13], and other publications.

In this paper we mostly focus on the study of Lipschitzian stability of solutions to variational systems governed by *parametric generalized equations*

$$0 \in f(x,y) + Q(x,y) \tag{1.1}$$

with the decision variable y and the parameter x, where $f\colon X \times Y \to Z$ is a single-valued mapping while $Q\colon X \times Y \rightrightarrows Z$ is a set-valued mappings between Banach spaces. For convenience we use the terms *base* and *field* referring to the single-valued and set-valued parts of (1.1), respectively.

Generalized equations were introduced by [26] as an extension of standard equations with no multivalued part. They have been widely recognized as a convenient model for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibrium, etc. In particular, generalized equations (1.1) reduce to parametric *variational inequalities*

$$\text{find} \quad y \in \Omega \quad \text{with} \quad \langle f(x,y), v - y \rangle \geqslant 0 \quad \text{for all} \quad v \in \Omega \tag{1.2}$$

when $Q(y) = N(y;\Omega)$ is the normal cone mapping generated by a convex set Ω. The classical *complementarity* problem correspond to (1.2) when Ω is the nonnegative orthant in IR^n. In contrast to the standard framework, we consider the case when the field Q in (1.1) and hence the set Ω in (1.2) may *depend on the perturbation parameter* x. The latter model is particularly convenient for describing stationary point maps and stationary point-multiplier maps in optimization problems with parameter-dependent constraints.

Our objective is to study *Lipschitzian stability* of the *solution map*

$$S(x) := \{ y \in Y \mid 0 \in f(x,y) + Q(x,y) \}$$

to (1.1) when (x,y) vary around the reference point $(\bar{x},\bar{y}) \in \operatorname{gph} S$. We pay the main attention to the concept of Lipschitzian behavior introduced by [1] under the name of "pseudo-Lipschitzian" multifunctions. In our opinion, it is better to use the term *Lipschitz-like* multifunctions referring to this kind of Lipschitzian behavior, which is indeed probably the most proper extension of the classical Lipschitzian property to set-valued mappings (while "pseudo" means "false"; cf. [27], where this property of multifunctions is called the Aubin property without specifying its Lipschitzian nature). It is well known that Aubin's Lipschitz-like property of an arbitrary mapping $F\colon X \rightrightarrows Y$ between Banach spaces is equivalent to metric regularity as well as to linear openness of its inverse $F^{-1}\colon Y \rightrightarrows X$. These properties play a fundamental role in variational analysis and its applications.

After presenting basic definitions and preliminary material in Section 2, we devote Section 3 to coderivative characterizations of the mentioned fundamental properties with evaluating the exact bounds of the corresponding moduli. For these purposes we need two kinds of coderivatives: *normal* and *mixed*; see below. As a by-product of such characterizations and the coderivative calculus, we obtain an infinite-dimensional extension of the recent result by [5] on the relationship between the exact regularity bound (regularity modulus) and the so-called *radius of metric regularity* that gives a measure of the extent to which a set-valued mapping can be perturbed before metric regularity is lost. The latter relates to the classical Eckart–Young theorem in numerical analysis as well as to the condition number theorems in nonlinear programming.

Section 4 is devoted to applications of coderivative criteria for the Lipschitz-like property and calculus rules for coderivatives to Lipschitzian stability of variational systems (1.1) and their specifications. A crucial role in these developments in infinite-dimensional spaces is played by the calculus of *sequential normal compactness*, which is not needed in finite dimensions.

Throughout the paper we use standard notation, with special symbols introduced where they are defined. Unless otherwise stated, all spaces considered are Banach whose norms are always denoted by $\|\cdot\|$. For any space X we consider its dual space X^* equipped with the weak* topology w^*, where $\langle\cdot,\cdot\rangle$ means the canonical pairing. For multifunctions $F\colon X \rightrightarrows X^*$ the expression

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \,\Big|\, \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\Big\}$$

signifies the *sequential* Painlevé-Kuratowski upper (outer) limit with respect to the norm topology in X and the weak* topology in X^*; $I\!N := \{1, 2, \ldots\}$. Recall that $F\colon X \rightrightarrows Y$ is *positively homogeneous* if $F(\alpha x) = \alpha F(x)$ for all $x \in X$ and $\alpha > 0$. The *norm* a positive homogeneous multifunction is defined by

$$\|F\| := \sup\big\{\|y\| \,\big|\, y \in F(x) \text{ and } \|x\| \leqslant 1\big\}. \tag{1.3}$$

2. Basic Definitions and Preliminaries

In this section we introduce the basic concepts and constructions of our study and present necessary preliminaries used in what follows.

We say that a set-valued mapping $F\colon X \rightrightarrows Y$ is *Lipschitz-like* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\ell \geqslant 0$ if there are neighborhood U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell\|x-u\| B_Y \text{ for all } x, u \in U, \tag{2.1}$$

where B_Y stands for the closed unit ball in Y. The infimum of all such moduli $\{\ell\}$ is called the *exact Lipschitzian bound* of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$. The mapping F is *metrically regular* around $(\bar{x}, \bar{y})$ with modulus $\mu > 0$

if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a number $\gamma > 0$ such that

$$\operatorname{dist}(x; F^{-1}(y)) \leqslant \mu \operatorname{dist}(y; F(x)) \text{ for all } x \in U$$
$$\text{and } y \in V \text{ with } (y; F(x)) \leqslant \gamma.$$

The infimum of all such moduli $\{\mu\}$, denoted by $\operatorname{reg} F(\bar{x}, \bar{y})$, is called the *exact regularity bound* of F around $(\bar{x}, \bar{y})$.

If $V = Y$ is (2.1), the above Aubin's Lipschitz-like property reduces to the local Lipschitz continuity of F around $\bar{x}$ with respect to the Pompieu-Hausdorff distance on 2^Y, and for single-valued mappings $F = f$: $X \to Y$ it agrees with the classical local Lipschitz continuity. For general set-valued mappings F the (local) Lipschitz-like property can be viewed as a localization of Lipschitzian behavior not only relative to a point of the domain but also relative to a particular point of the image $\bar{y} \in F(\bar{x})$.

It is well known that F is Lipschitz-like around $(\bar{x}, \bar{y})$ with modulus $\ell > 0$ *if and only if* its inverse F^{-1} is metrically regular around $(\bar{y}, \bar{x})$ with the same modulus. Hence one always has

$$\operatorname{reg} F(\bar{x}, \bar{y}) = \operatorname{lip} F^{-1}(\bar{y}, \bar{x}). \tag{2.2}$$

We are able to provide complete dual characterizations of the Lipschitz-like and metric regularity properties using appropriate constructions of generalized differentiation. To present them in the next section, we first recall the definitions of coderivatives for set-valued mappings, which are the basic constructions of our study. The reader can consult [14] for more references and discussions.

Given F: $X \rightrightarrows Y$ and $\varepsilon \geqslant 0$, define the *ε-coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ as the set-mapping $\hat{D}^*_\varepsilon F(\bar{x}, \bar{y})$: $Y^* \rightrightarrows X^*$ with the values

$$\hat{D}^*_\varepsilon F(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* \middle| \limsup_{(x,y) \overset{gphF}{\to} (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle}{\|x - \bar{x}\| + \|y - \bar{y}\|} \leqslant \varepsilon \right\}, \quad y^* \in Y^*, \tag{2.3}$$

where $x \overset{\Omega}{\to} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. We put $\hat{D}^*_\varepsilon F(\bar{x}, \bar{y})(y^*) = \emptyset$ for all $y^* \in Y^*$ and $\varepsilon \geqslant 0$.

If $(\bar{x}, \bar{y}) \notin \operatorname{gph} F$, and $\widehat{D}^* F(\bar{x}, \bar{y}) := \widehat{D}^*_0 F(\bar{x}, \bar{y})$. Then the *normal coderivative* of F at $(\bar{x}, \bar{y})$ is defined by

$$D^*_N F(\bar{x}, \bar{y})(\bar{y}^*) := \mathop{\operatorname{Lim\,sup}}_{\substack{(x,y) \to (\bar{x}, \bar{y}) \\ Y^* \overset{w^*}{\to} \bar{y}^* \\ \varepsilon \downarrow 0}} \widehat{D}^*_\varepsilon F(x, y)(y^*), \tag{2.4}$$

that is, $\bar{x}^* \in D^*_N F(\bar{x}, \bar{y})(\bar{y}^*)$ if and only if there are sequences $\varepsilon_k \downarrow 0$, $(x_k, y_k) \to (\bar{x}, \bar{y})$, and $(x^*_k, y^*_k) \overset{w^*}{\to} (\bar{x}^*, \bar{y}^*)$ with $(x_k, y_k) \in \operatorname{gph} F$ and $x^*_k \in \hat{D}^*_{\varepsilon_k} F(x_k, y_k)(y^*_k)$. The *mixed coderivative* of F at $(\bar{x}, \bar{y})$ is defined by

$$D^*_M F(\bar{x}, \bar{y})(\bar{y}^*) := \mathop{\operatorname{Lim\,sup}}_{\substack{(x,y) \to (\bar{x}, \bar{y}) \\ Y^* \to \bar{y}^* \\ \varepsilon \downarrow 0}} \hat{D}^*_\varepsilon F(x, y)(y^*), \tag{2.5}$$

that is, $D^*_M F(\bar{x},\bar{y})(\bar{y}^*)$ is the collection of such $\bar{x}^*\in X^*$ for which there are sequences $\varepsilon_k\downarrow 0$, $(x_k,y_k,y^*_k)\to(\bar{x},\bar{y},\bar{y}^*)$, and $x^*_k\xrightarrow{w^*}\bar{x}^*$ with $(x_k,y_k)\in\operatorname{gph}F$ and $x^*_k\in\hat{D}^*_{\varepsilon_k}F(x_k,y_k)(y^*_k)$. One can equivalently put $\varepsilon=0$ in (2.4) and (2.5) if F is closed-graph around $(\bar{x},\bar{y})$ and if both X and Y are *Asplund*, i.e., such Banach spaces on which every convex continuous function is generically Fréchet differentiable (in particular, any reflexive spaces); see [24] for more information on Asplund spaces.

It follows from the definitions that $D^*_M F(\bar{x},\bar{y})(y^*)\subset D^*_N F(\bar{x},\bar{y})(y^*)$ when the equality obviously holds if Y is finite-dimensional. Note that the above inclusion may be strict even for single-valued Lipschitzian mappings $f\colon I\!R\to Y$ with values in Hilbert spaces Y that are Fréchet differentiable at $\bar{x}$. The class of mappings for which $D^*_N F(\bar{x},\bar{y})=D^*_M F(\bar{x},\bar{y})$ plays an important role in the results presented in the next section. This happens, in particular, when F is *graphically regular* at $(\bar{x},\bar{y})$ in the sense that

$$D^*_N F(\bar{x},\bar{y})(y^*)=\hat{D}^*F(\bar{x},\bar{y})(y^*)\ \text{for all}\ y^*\in Y^*.$$

The latter class includes, in particular, set-valued mappings with convex graphs and also single-valued mappings strictly differentiable at $\bar{x}$ for which

$$\hat{D}^*f(\bar{x})(y^*)=D^*_N f(\bar{x})(y^*)=D^*_M f(\bar{x})(y^*)=\{\nabla f(\bar{x})^*y^*\},\quad y^*\in Y^*. \tag{2.6}$$

On the other hand, a single-valued Lipschitzian mapping between finite-dimensional spaces is never graphically regular unless it is strictly differentiable at a reference point.

Given an extended-real-valued function $\varphi\colon X\to\overline{I\!R}:=[-\infty,\infty]$ finite at $\bar{x}$, we consider its (first-order) *subdifferential* at $\bar{x}$ defined by

$$\partial\varphi(\bar{x}):=D^*_N E_\varphi(\bar{x},\varphi(\bar{x}))(1)\ \text{with}\ E_\varphi(x):=\{\nu\in I\!R\,|\,\nu\geqslant\varphi(x)\}. \tag{2.7}$$

There are intrinsic representations of $\partial\varphi$ and complete subdifferential theories, which can be found in [27] in finite-dimensional spaces and in [17] in infinite dimensions. If $f\colon X\to Y$ is Lipschitz continuous around $\bar{x}$, then its coderivatives (2.4) and (2.5) are related to the subdifferential (2.7) via the scalarization formulas:

$$D^*_M f(\bar{x})(y^*)=\partial\langle y^*,f\rangle(\bar{x}),\quad D^*_N f(\bar{x})(y^*)=\partial\langle y^*,f\rangle(\bar{x}),\quad y^*\in Y^*, \tag{2.8}$$

where the first formula holds in any Banach spaces, while the second one requires that X is Asplund and f is w^**-strictly Lipschitzian* around $\bar{x}$ in the following sense: f is Lipschitz continuous around $\bar{x}$ and for every $v\in X$ and every sequences $x_k\to\bar{x}$, $t_k\downarrow 0$ and $y^*_k\xrightarrow{w^*}0$ as $k\to\infty$ one has

$$\langle y^*_k,y_k\rangle\to 0,\ \text{where}\ y_k:=\frac{f(x_k+t_k v)-f(x_k)}{t_k},\quad k\in\mathbb{N}.$$

The latter property always holds when f is compactly Lipschitzian in the sense of [28].

The coderivative and subdifferential constructions (2.4), (2.5), and (2.7) enjoy fairly rich calculi in both finite-dimensional and infinite-dimensional settings; see [27] and [14] with the references therein. These calculi require natural qualification conditions and also the so-called "normal compactness" conditions needed only in infinite dimensions; see [2, 6, 7, 19, 23] for the genesis of such properties and various applications. The following two properties formulated in [18] are of particular interest for applications in this paper.

A mapping $F\colon X \rightrightarrows Y$ is *sequentially normally compact* (SNC) at $(\bar{x},\bar{y})\in \operatorname{gph} F$ if for any sequences $(\varepsilon_k, x_k, y_k, x_k^*, y_k^*)\in[0,\infty)\times(\operatorname{gph} F)\times X^*\times Y^*$ satisfying

$$\varepsilon_k \downarrow 0,\ (x_k, y_k)\to(\bar{x},\bar{y}),\ x_k^*\in \hat{D}^*_{\varepsilon_k}F(x_k,y_k)(y_k^*) \tag{2.9}$$

one has $(x_k^*, y_k^*)\xrightarrow{w^*}(0,0)\Longrightarrow \|(x_k^*,y_k^*)\|\to 0$ as $k\to\infty$. A mapping F is *partially sequentially normally compact* (PSNC) at $(\bar{x},\bar{y})$ if for any above sequences satisfying (2.9) one has

$$\Big[x_k^*\xrightarrow{w^*}0 \text{ and } \|y_k^*\|\to 0\Big]\Longrightarrow \|x_k^*\|\to 0 \text{ as } k\to\infty.$$

We may equivalently put $\varepsilon_k=0$ in the above properties if both X and Y are Asplund while F is closed-graph around $(\bar{x},\bar{y})$. Finally, a set $\Omega\subset X$ is SNC at $\bar{x}\in\Omega$ if the constant mapping $F(x)\equiv\Omega$ satisfies this property.

Note that the SNC property of sets and mappings are closely related to the compactly epi-Lipschitzian property of [2]; see [6] and [8] on recent results in this direction. For closed convex sets $\Omega\subset X$ the latter property holds if and only if the affine hull of Ω is a closed finite-codimensional subspace of X with $\operatorname{ri}\Omega\neq\emptyset$; cf. [3]. On the other hand, every Lipschitz-like mapping $F\colon X\rightrightarrows Y$ between Banach spaces is PSNC at $(\bar{x},\bar{y})$, and hence it is SNC at this point when $\dim Y<\infty$; see Theorem 3.3 in the next section. We refer the reader to [21] for an extended calculus involving SNC and PSNC properties applied below.

3. Coderivative Characterizations of Lipschitzian Stability and Metric Regularity

We start with point-based characterizations of Lipschitzian behavior for infinite-dimensional multifunctions. The main criterion gives necessary and sufficient conditions for the Lipschitz-like property of F around $(\bar{x},\bar{y})$ in terms of the mixed coderivative $D^*_M F(\bar{x},\bar{y})$ and the PSNC property of F at $(\bar{x},\bar{y})$, while the principal upper estimate of the exact Lipschitz bound $\operatorname{lip} F(\bar{x},\bar{y})$ is expressed via the normal coderivative $D^*_N F(\bar{x},\bar{y})$. This implies the *precise formula* for computing the exact bound $\operatorname{lip} F(\bar{x},\bar{y})$ for set-valued mappings satisfying the following requirements that link the two coderivatives. Note that norms of the coderivatives, as positively homogeneous multifunctions from Y^* into X^*, are computed by formula (1.3).

DEFINITION 3.1. Let $F\colon X \rightrightarrows Y$ be a set-valued mapping between Banach spaces, and let $(\bar{x},\bar{y})\in \operatorname{gph} F$. Then we say that:

(i) F is *coderivatively normal* at $(\bar{x},\bar{y})$ if $\|D^*_M F(\bar{x},\bar{y})\| = \|D^*_N F(\bar{x},\bar{y})\|$.
(ii) F is *strongly coderivative normal* at $(\bar{x},\bar{y})$ if

$$D^*_M F(\bar{x},\bar{y}) = D^*_N F(\bar{x},\bar{y}) := D^* F(\bar{x},\bar{y}).$$

Note that coderivative normality may not always hold even for single-valued Lipschitz continuous mappings from $I\!R$ into a Hilbert space that are Fréchet differentiable at the point of interest; see Example 2.9 from [20], where $\|D^*_M f(0)\| = 0$ while $\|D^*_N f(0)\| = \infty$. The next proposition lists some important classes of mappings that are strongly coderivatively normal (and hence coderivatively normal) at reference points.

PROPOSITION 3.2. *$F\colon X \rightrightarrows Y$ is strongly coderivatively normal at $(\bar{x},\bar{y})$ if it satisfies one of the following conditions*:

(a) *Y is finite-dimensional.*

(b) *F is graphically regular at $(\bar{x},\bar{y})$.*

(c) *F is single-valued and w^*-strictly Lipschitzian around $\bar{x}$, and X is Asplund.*

(d) *F is the indicator mapping of a set $\Omega\subset X$ relative to Y, i.e., $F(x)=0\in Y$ if $x\in\Omega$ and $F(x)=\emptyset$ otherwise.*

(e) *$F=f\circ g$, where $g\colon X\to I\!R^n$ is Lipschitz continuous around $\bar{x}$ and $f\colon I\!R^n\to Y$ is strictly differentiable at $g(\bar{x})$.*

(f) *$F=f+F_1$, where $f\colon X\to Y$ is strictly differentiable at $\bar{x}$ and $F_1\colon X\rightrightarrows Y$ is strongly coderivatively normal at $(\bar{x},\bar{y}-f(\bar{x}))$.*

(g) *$F=F_1\circ g$, where $g\colon X\to Z$ is strictly differentiable at $\bar{x}$ with the surjective derivative and where $F_1\colon Z\rightrightarrows Y$ is strongly coderivatively normal at $(g(\bar{x}),\bar{y})$.*

(h) *$F=\partial(\varphi\circ g)$, where $\varphi\colon Z\to \overline{I\!R}$ and $g\in C^2$ with the surjective derivative $\nabla g(\bar{x})$, where either $\nabla g(\bar{x})^*$ is complemented in X^* or the closed unit ball of X^{**} is weak* sequentially compact (the latter is automatic when either X is reflexive or X^* is separable), and where $\partial\varphi$ is strongly coderivatively normal at $(\bar{z},\bar{v})$ with $\bar{z}:=g(\bar{x})$ and $\bar{v}$ uniquely defined by*

$$\bar{y}=\nabla g(\bar{x})^*\bar{v} \quad \textit{and} \quad \bar{v}\in\partial\varphi(\bar{z}).$$

Proof. Properties (a)–(c) have been discussed in Section 2; (d) is elementary; (e) and (f) follow from the coderivative calculus in [14]; (g) and (h) are proved in [22]. □

As the main tool of our analysis we will use the following coderivative characterizations of Lipschitzian behavior of multifunctions given in [14]; see also the references therein.

THEOREM 3.3. *Let $F\colon X\rightrightarrows Y$ be closed-graph around $(\bar{x},\bar{y})$. Consider the properties*:

(a) *F is Lipschitz-like around $(\bar{x},\bar{y})$.*

(b) *F is PSNC at $(\bar{x},\bar{y})$ and $\|D^*_M F(\bar{x},\bar{y})\|<\infty$.*

(c) *F is PSNC at* $(\bar{x},\bar{y})$ *and* $D^*_M F(\bar{x},\bar{y})(0)=\{0\}$.
Then (a)⇒(b)⇒(c) *while these properties are equivalent if both X and Y are Asplund. Moreover, one has the estimates*

$$\|D^*_M F(\bar{x},\bar{y})\| \leqslant \operatorname{lip} F(\bar{x},\bar{y}) \leqslant \|D^*_N F(\bar{x},\bar{y})\|$$

for the exact Lipschitzian bound of F around $(\bar{x},\bar{y})$, *where the upper estimate holds if* $\dim X<\infty$ *and Y is Asplund. Thus*

$$\operatorname{lip} F(\bar{x},\bar{y}) = \|D^*_M F(\bar{x},\bar{y})\| = \|D^*_N F(\bar{x},\bar{y})\| \tag{3.1}$$

if in addition F is coderivatively normal at $(\bar{x},\bar{y})$.

If both X and Y are finite-dimensional, then F is automatically PSNC and coderivatively normal at $(\bar{x},\bar{y})$, and we get the coderivative criterion for the Aubin Lipschitz-like property

$$D^*F(\bar{x},\bar{y})(0)=\{0\} \quad \text{with} \quad \operatorname{lip} F(\bar{x},\bar{y}) = \|D^*F(\bar{x},\bar{y})\|$$

from [11]; see also Theorem 9.40 in [27] with the references and commentaries therein.

In the next two sessions we present applications of Theorem 3.3 to Lipschitzian stability of variational systems. Now let us employ this theorem to an infinite-dimensional extension of the result in [5] on computing the *radius of metric regularity* of $F\colon X \rightrightarrows Y$ at $(\bar{x},\bar{y})$ defined by

$$\operatorname{rad} F(\bar{x},\bar{y}) := \inf_{g\in L(X,Y)} \Big\{ \|g\| \,\Big|\, F+g \text{ not metrically regular around } \big(\bar{x},\bar{y}+g(\bar{x})\big) \Big\},$$

where $L(X,Y)$ denotes the space of linear continuous operators from X into Y.

THEOREM 3.4. *Let* $F\colon X\rightrightarrows Y$ *be closed-graph around some point* $(\bar{x},\bar{y})\in \operatorname{gph} F$. *Assume that X is Asplund, that* $\dim Y<\infty$, *and that* F^{-1} *is coderivatively normal at* $(\bar{x},\bar{y})$. *Then one has*

$$\operatorname{rad} F(\bar{x},\bar{y}) = 1\big/\operatorname{reg} F(\bar{x},\bar{y}). \tag{3.2}$$

Furthermore, under these assumptions the infimum in the definition of rad $F(\bar{x},\bar{y})$ *is unchanged if taken with respect to* $g\in L(X,Y)$ *of rank one, but also is unchanged when the space of perturbations g is enlarged from linear operators to locally Lipschitzian mappings*:

$$\operatorname{rad} F(\bar{x},\bar{y}) = \inf_{g\colon X\to Y} \Big\{ \operatorname{lip} g(\bar{x}) \,\Big|\, F+g \textit{ not metrically regular around } (\bar{x},\bar{y}+g(\bar{x})) \Big\}. \tag{3.3}$$

Proof. It has been proved in [5] that for every closed-graph mapping $F\colon X\rightrightarrows Y$ between Banach spaces one has

$$\inf_{g\colon X\to Y} \Big\{ \operatorname{lip} g(\bar{x}) \,\Big|\, F+g \text{ not metrically regular around } (\bar{x},\bar{y}+g(\bar{x})) \Big\} \geqslant 1\big/\operatorname{reg} F(\bar{x},\bar{y}). \tag{3.4}$$

This immediately implies the inequality "$\geqslant$" in (3.2). Moreover, (3.4) ensures, since $\mathrm{lip}\, g(\bar{x}) = \|g\|$ for linear continuous mappings g, that (3.2) implies (3.3). Thus it remains to prove that (3.2) holds under the assumptions made, and the infimum in the definition of $\mathrm{rad}\, F(\bar{x}, \bar{y})$ is unchanged when restricted to linear operators $g \in L(X, Y)$ of rank one. We are going to furnish this, following the line in [5], with the usage of Theorem 3.3 and coderivative calculus in infinite-dimensional spaces.

Due to the equivalence (2.2) between metric regularity of set-valued mappings and the Lipschitz-like property of their inverse, we apply Theorem 3.3 to inverse mappings. Note that for any $G\colon X \rightrightarrows Y$ with $\dim Y < \infty$ the PSNC property of G^{-1} holds *automatically*. Hence, by (a)$\Leftrightarrow$(b) in Theorem 3.3 applied to $(F+g)^{-1}\colon Y \rightrightarrows X$ when X is Asplund and Y is finite-dimensional, we conclude that $F+g$ is *not* metrically regular around $(\bar{x}, \bar{y}+g(\bar{x}))$ *if and only if*

$$\|D_M^*(F+g)^{-1}(\bar{y}+g(\bar{x}), \bar{x})\| = \|\widetilde{D}_M^*(F+g)(\bar{x}, \bar{y}+g(\bar{x}))^{-1}\| = \infty, \tag{3.5}$$

where $\widetilde{D}_M^* G(\bar{x}, \bar{y})(y^*) := \big\{x^* \in X^* \big| \, y^* \in -D_M^* G^{-1}(\bar{y}, \bar{x})(-x^*)\big\}$. Let us show that

$$\widetilde{D}_M^*(F+g)(\bar{x}, \bar{y}+g(\bar{x}))(y^*) = \widetilde{D}_M^* F(\bar{x}, \bar{y})(y^*) + g^*(y^*), \quad g \in L(X, Y), \tag{3.6}$$

provided that Y is finite-dimensional (the latter actually holds for any $g\colon X \to Y$ strictly differentiable at $\bar{x}$ with the replacement of g in (3.6) by $\nabla g(\bar{x})$). Indeed, taking $x^* \in \widetilde{D}_M^*(F+g)(\bar{x}, \bar{y}+g(\bar{x}))(y^*)$ and using the construction of $\widetilde{D}_M^*$ in Asplund spaces as well as $\dim Y < \infty$, we find sequences $x_k \to \bar{x}$, $y_k \to \bar{y}$ with $y_k \in F(x_k)$, and $(x_k^*, y_k^*) \to (x^*, y^*)$ such that $x_k^* \in \hat{D}^*(F+g)(x_k, y_k+g(x_k))$ for all $k \in \mathbb{N}$. Using the elementary calculus rule

$$\hat{D}^*(F+g)(x_k, y_k+g(x_k))(y_k^*) = \hat{D}^* F(x_k, y_k)(y_k^*) + g^*(y_k^*),$$

we get $x_k^* - g^*(y_k^*) \in \hat{D}^* F(x_k, y_k)(y_k^*)$. Since $x_k^* - g^*(y_k^*) \to x^* - g^*(y^*)$, the latter gives by passing to the limit as $k \to \infty$ that $x^* \in \widetilde{D}_M^* F(\bar{x}, \bar{y})(y^*) + g^*(y^*)$, which proves the inclusion"$\subset$" in (3.6). The opposite inclusion in (3.6) follows from

$$\widetilde{D}_M^*\big[(F+g)+(-g)\big](\bar{x}, \bar{y})(y^*) \subset \widetilde{D}_M^*(F+g)(\bar{x}, \bar{y}+g(\bar{x}))(y^*) - g^*(y^*).$$

Thus (3.5) is equivalent to

$$\|(\widetilde{D}_M^* F(\bar{x}, \bar{y}) + g^*)^{-1}\| = \infty, \quad g \in L(X, Y).$$

Now applying the exact bound formula (3.1) of Theorem 3.3 to the mapping F^{-1} that is assumed to be coderivatively normal at $(\bar{x}, \bar{y})$ and taking into account that $\|g^*\| = \|g\|$ for $g \in L(X, Y)$, we identify the targeted equality (3.2) with

$$\inf_{g \in L(X,Y)} \Big\{ \|g^*\| \,\Big|\, \|\widetilde{D}_M^* F(\bar{x}, \bar{y}) + g^*\| = \infty \Big\} = 1 \big/ \|\widetilde{D}_M^* F(\bar{x}, \bar{y})^{-1}\|. \tag{3.7}$$

Observe that every $h \in L(Y^*, X^*)$ can be represented as the adjoint operator $g^*\colon Y^* \to X^*$ for some $g \in L(X, Y)$ provided that Y is reflexive (in our case

$\dim Y < \infty$). Indeed, since $X \subset X^{**}$ and $Y^{**} = Y$, we construct $g \in L(X,Y)$ as the restriction on X of h^*: $X^{**} \to Y^{**}$. From this we can conclude that (3.7) is a special case of the extended Eckart–Young theorem (Theorem 2.6 in [5]) applied to the positive homogeneous mapping $\widetilde{D}^*_M F(\bar{x},\bar{y})$: $Y^* \rightrightarrows X^*$. This justifies (3.2) and ends the proof of the theorem. □

4. Lipschitzian Stability of Variational Systems

In this section we obtain sufficient conditions, as well as necessary and sufficient conditions, for the Lipschitz-like property of the solution maps

$$S(x) := \left\{ y \in Y \,\middle|\, 0 \in f(x,y) + Q(x,y) \right\} \tag{4.1}$$

to variational systems governed by parametric generalized equations (1.1). We are going to do it via the coderivative characterizations of Theorem 3.3, namely, applying the criteria in (c). The first step is to obtain coderivative representations for the solution map (4.1) in terms of the initial data of (1.1). This is the contents of the next theorem, the proof of which is based on the coderivative calculus in Asplund as well as general Banach spaces.

THEOREM 4.1. *Let* $(\bar{x},\bar{y})$ *satisfy* (1.1), *where* X, Y, Z *are Asplund,* f: $X \times Y \to Z$ *is continuous around* $(\bar{x},\bar{y})$, *and the graph of* Q *is closed around* $(\bar{x},\bar{y},\bar{z})$ *with* $\bar{z} := -f(\bar{x},\bar{y})$. *Then*

$$\begin{aligned} D^*S(\bar{x},\bar{y})(y^*) \subset \Big\{ x^* \in X^* \,\Big|\, &\exists z^* \in Z^* \text{ with} \\ &(x^*, -y^*) \in D^*_N f(\bar{x},\bar{y})(z^*) + D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*) \Big\} \end{aligned} \tag{4.2}$$

for both coderivatives $D^* = D^*_N, D^*_M$ *of the solution map* (4.1) *at* $(\bar{x},\bar{y})$ *provided that either one of the following conditions holds*:

(a) Q *is SNC at* $(\bar{x},\bar{y},\bar{z})$ *and*

$$\begin{aligned} \Big[(x^*, y^*) \in D^*_N f(\bar{x},\bar{y})(z^*) \cap \big(-D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*)\big)\Big] &\Longrightarrow \\ (x^*, y^*, z^*) = (0,0,0), \end{aligned} \tag{4.3}$$

which is equivalent to

$$\Big[0 \in \partial\langle z^*, f\rangle(\bar{x},\bar{y}) + D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*)\Big] \Longrightarrow z^* = 0 \tag{4.4}$$

if f *is* w^**-strictly Lipschitzian around* $(\bar{x},\bar{y})$.

(b) f *is Lipschitz continuous around* $(\bar{x},\bar{y})$, $\dim Z < \infty$, *and* (4.4) *is satisfied.*

If in addition to either (a) *or* (b) f *is strictly differentiable at* $(\bar{x},\bar{y})$ *and* Q *is graphically regular at* $(\bar{x},\bar{y},\bar{z})$, *then* S *is graphically regular at* $(\bar{x},\bar{y})$ *and* (4.2) *holds as equality.*

Moreover, one has the equality

$$D_N^* S(\bar{x},\bar{y})(y^*)=\Big\{x^*\in X^*\Big|\ \exists z^*\in Z^* \text{ with } x^*=\nabla_x f(\bar{x},\bar{y})^* z^*,\\ -y^*\in\nabla_y f(\bar{x},\bar{y})^* z^*+D_N^* Q(\bar{y},\bar{z})(z^*)\Big\} \tag{4.5}$$

in general Banach spaces X,Y,Z without the above assumptions if $\nabla_x f(\bar{x},\bar{y})$ is surjective and Q does not depend on x.

Proof. Observe that the graph of the solution map S in (4.1) is represented as

$$\text{gph} S=\big\{(x,y)\in X\times Y\big|\, g(x,y)\in\Theta\big\}:=g^{-1}(\Theta) \text{ with } \Theta:=\text{gph} Q, \tag{4.6}$$

where one has

$$g(x,y):=\big(x,y,-f(x,y)\big),\quad \text{gph} Q\subset X\times Y\times Z \text{ if } Q=Q(x,y); \tag{4.7}$$

$$g(x,y):=\big(y,-f(x,y)\big),\quad \text{gph} Q\subset Y\times Z \text{ if } Q=Q(y). \tag{4.8}$$

It is sufficient to prove (4.2) for the normal coderivative $D^*S=D_N^*S$, since the mixed one is always smaller. As well known, the normal coderivative admits the representation

$$D_N^* S(\bar{x},\bar{y})(y^*)=\big\{x^*\in X^*\big|\,(x^*,-y^*)\in N((\bar{x},\bar{y});\text{gph} S)\big\} \tag{4.9}$$

in terms of the *normal cone* $N(\bar{w};\Omega):=\partial\delta(\bar{w};\Omega)$ to Ω at $\bar{w}\in\Omega$ defined via the subdifferential (2.7) of the indicator function. Taking this into account, we employ the calculus rule

$$N((\bar{x},\bar{y});\text{gph} S)\subset D_N^* g(\bar{x},\bar{y})\circ N(\bar{z};\Theta) \text{ with } \Theta=\text{gph} Q \tag{4.10}$$

held under the qualification condition

$$N(\bar{z};\Theta)\cap\ker D_N^* g(\bar{x},\bar{y})=\{0\} \tag{4.11}$$

provided that either Θ is SNC at $\bar{z}$ or g^{-1} is PSNC at $(\bar{z},\bar{x},\bar{y})$; see Theorem 4.5 in [14]. Moreover, the equality holds in (4.10) if g is strictly differentiable at $(\bar{x},\bar{y})$ and Q is graphically regular at $(\bar{x},\bar{y},\bar{z})$. Since

$$g(x,y)=(x,y,0)+(0,0,-f(x,y))$$

for g in (4.7) and $D_N^*(-f)(\bar{x},\bar{y})(z^*)=D_N^* f(\bar{x},\bar{y})(-z^*)$, one has

$$D_N^* g(\bar{x},\bar{y})(x^*,y^*,z^*)=(x^*,y^*)+D_N^* f(\bar{x},\bar{y})(-z^*)$$

by an elementary sum rule for coderivatives. Then it is easy to check that the qualification condition (4.11) is equivalent to (4.3), which reduces to (4.4) for w^*-strict Lipschitzian mappings due to the second scalarization formula in (2.8). Similarly we check that (4.9) and (4.10) imply the required coderivative inclusion (4.2). This proves the theorem under the assumptions in (a).

To prove the theorem under the assumptions in (b), it remains to show that the PSNC property of g^{-1} holds if f is Lipschitz continuous around $(\bar{x},\bar{y})$ while Z is finite-dimensional. By the structure of g in (4.7) and the mentioned scalarization formula we conclude that the PSNC property of g^{-1} at $(\bar{w},\bar{x},\bar{y})$ means in this setting that for every sequences $(x_k,y_k)\to(\bar{x},\bar{y})$, $(u_k^*,v_k^*)\xrightarrow{w^*}(0,0)$, and

$$(x_k^*,y_k^*)-(u_k^*,v_k^*)\in\hat{\partial}\langle -z_k^*,f\rangle(x_k,y_k)\ \text{ with }\ \|(x_k^*,y_k^*,z_k^*)\|\to 0$$

one has $\|(u_k^*,v_k^*)\|\to 0$ as $k\to\infty$, where

$$\hat{\partial}\varphi(\bar{x}):=\left\{x^*\in X^*\,\middle|\,\frac{\varphi(x)-\varphi(\bar{x})-\langle x^*,x-\bar{x}\rangle}{\|x-\bar{x}\|}\geqslant 0\right\} \tag{4.12}$$

for $\varphi\colon X\to\overline{IR}$. The latter statement statement easily follows from (4.12).

Finally, let us prove equality (4.5) when $Q=Q(y)$ and $f\colon X\times Y\to Z$ is a mapping between Banach spaces strictly differentiable at $(\bar{x},\bar{y})$ and such that $\nabla_x f(\bar{x},\bar{y})$ is surjective. It is easy to see that the surjectivity of $\nabla_x f(\bar{x},\bar{y})$ is equivalent to the surjectivity of $\nabla g(\bar{x},\bar{y})$ for g defined in (4.8).

To proceed, we mention that the *equality* holds in (4.10) without any assumptions on Θ in Banach spaces provided that g is strictly differentiable at $(\bar{x},\bar{y})$ with surjective derivative; see [22]. This implies (4.5) by computing $\nabla g(\bar{x},\bar{y})$ in (4.8) via representation (4.9) and elementary calculations. □

Now we are ready to obtain coderivative conditions for Lipschitzian stability of the solution map S in (4.1) via the general characterizations of Theorem 3.3, coderivative representations of Theorem 4.1, and appropriate results of the SNC calculus. We present two theorems in this direction. The first one contains *necessary and sufficient* conditions for Lipschitzian stability with computing the exact Lipschitzian bound of the solution map.

THEOREM 4.2. *Let $f\colon X\times Y\to Z$ be strictly differentiable at $(\bar{x},\bar{y})\in\operatorname{gph}S$ in (4.1), let $Q\colon X\times Y\rightrightarrows Z$ be locally closed-graph around $(\bar{x},\bar{y},\bar{z})$ with $\bar{z}=-f(\bar{x},\bar{y})$ and SNC at this point, and let X,Y be Asplund. The following hold*:

(i) *Assume that Z is Banach, that $\nabla_x f(\bar{x},\bar{y})$ is surjective, and that $Q=Q(y)$. Then S is Lipschitz-like around $(\bar{x},\bar{y})$ if*

$$\left[0\in\nabla_y f(\bar{x},\bar{y})^*z^*+D_N^*Q(\bar{x},\bar{y},\bar{z})(z^*)\right]\Longrightarrow z^*=0. \tag{4.13}$$

This condition is also necessary for the Lipschitz-like property of S around $(\bar{x},\bar{y})$ if S is strongly coderivatively normal at $(\bar{x},\bar{y})$, in particular, when $\dim Y<\infty$. *If in addition* $\dim X<\infty$, *then*

$$\operatorname{lip}S(\bar{x},\bar{y})=\sup\Big\{\|\nabla_x f(\bar{x},\bar{y})^*z^*\|\ \Big|\ \exists y^*\in D_N^*Q(\bar{y},\bar{z})(z^*)\ \textit{with}\ \|\nabla_y f(\bar{x},\bar{y})^*z^*+y^*\|\leqslant 1\Big\}.$$

(ii) *Assume that Z is Asplund and that Q is graphically regular at $(\bar{x},\bar{y},\bar{z})$. Then S is also graphically regular at $(\bar{x},\bar{y})$, and the condition*

$$\Big[(x^*,0)\in\nabla f(\bar{x},\bar{y})^*z^*+D^*Q(\bar{x},\bar{y},\bar{z})(z^*)\Big]\Longrightarrow x^*=z^*=0 \tag{4.14}$$

is sufficient for the Lipschitz-like property of S around $(\bar{x},\bar{y})$. This condition is also necessary for the Lipschitz-like property of S provided that

$$\Big[0\in\nabla f(\bar{x},\bar{y})^*z^*+D^*Q(\bar{x},\bar{y},\bar{z})(z^*)\Big]\Longrightarrow z^*=0.$$

If in addition $\dim X<\infty$, *then*

$$\operatorname{lip}S(\bar{x},\bar{y})=\sup\Big\{\|x^*\|\,\Big|\,\exists z^*\in Z^*\ \textit{with}\ \Big(x^*-\nabla_x f(\bar{x},\bar{y})^*z^*,\\ -y^*-\nabla_y f(\bar{x},\bar{y})^*z^*\Big)\in D^*Q(\bar{x},\bar{y},\bar{z})(z^*),\ \|y^*\|\leqslant 1\Big\}.$$

Proof. Using Theorem 4.1 and the coderivative representations (2.6) for strict differentiable mappings, one can check that the qualification conditions (4.13) and (4.14) ensure the fulfillment of $D^*_M S(\bar{x},\bar{y})(0)=\{0\}$; moreover, these conditions are also necessary for the latter criterion under the additional assumptions in (i) and (ii), respectively. Similarly we derive the above formulas for $\operatorname{lip}S(\bar{x},\bar{y})$ from (3.1) and Theorem 4.1. To complete the proof of the theorem, it suffices to show that S is SNC at $(\bar{x},\bar{y})$ under the assumptions made.

We have mentioned that in case (i) the surjectivity of $\nabla_x f(\bar{x},\bar{y})$ is equivalent to the surjectivity of $\nabla g(\bar{x},\bar{y})$ with g defined in (4.8). Since $\operatorname{gph}S=g^{-1}(\operatorname{gph}Q)$, we conclude from here that the SNC property of S at $(\bar{x},\bar{y})$ is equivalent to the one for Q assumed in the theorem; see [22] for more details. Under the assumptions in (b) the SNC property of S at $(\bar{x},\bar{y})$ follows from Theorem 5.8 in [21]. □

Finally, we give sufficient conditions of Lipschitzian stability with upper bound estimates for nonsmooth and nonregular variational systems.

THEOREM 4.3. *Let $f\colon X\times Y\to Z$ be a mapping between Asplund spaces that is continuous around $(\bar{x},\bar{y})\in\operatorname{gph}S$ in (4.1), and let $Q\colon X\times Y\rightrightarrows Z$ be locally closed-graph around $(\bar{x},\bar{y},\bar{z})$ with $\bar{z}=-f(\bar{x},\bar{y})$ and SNC at this point. Assume that f is PSNC at $(\bar{x},\bar{y})$ (which is automatic when f is locally Lipschitzian) and that one has the qualification conditions*

$$\Big[(x^*,0)\in D^*_N f(\bar{x},\bar{y})(z^*)+D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*)\Big]\Longrightarrow x^*=0,$$
$$\Big[(x^*,y^*)\in D^*_N f(\bar{x},\bar{y})(z^*)\cap\big(-D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*)\big)\Big]\Longrightarrow x^*=y^*=z^*=0,$$

which together are equivalent to

$$\Big[(x^*,0)\in\partial\langle z^*,f\rangle(\bar{x},\bar{y})+D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*)\Big]\Longrightarrow x^*=z^*=0 \tag{4.15}$$

when f is w^*-strictly Lipschitzian around $(\bar{x},\bar{y})$. Then S is Lipschitz-like around this point. If in addition $\dim X<\infty$, *then*

$$\operatorname{lip} S(\bar{x},\bar{y})\leqslant \sup\Big\{\|x^*\|\ \Big|\ \exists z^*\in Z^* \text{ with } (x^*,-y^*)\in\partial\langle z^*,f\rangle(\bar{x},\bar{y}) \\ +D_N^*Q(\bar{x},\bar{y},\bar{z})(z^*),\ \|y^*\|\leqslant 1\Big\}.$$

Proof. Observe that the assumptions made in this theorem imply the fulfillment of all the assumptions in Theorem 4.1. Hence the coderivative inclusion (4.2) holds, and thus the first qualification condition of the theorem ensures that $D_M^*S(\bar{x},\bar{y})(0)=\{0\}$. By Theorem 3.1 it remains to show that S is PSNC at $(\bar{x},\bar{y})$.

Let us prove that S is actually SNC at $(\bar{x},\bar{y})$ if f is PSNC at this point in addition to the qualification conditions and the SNC property of Q at $(\bar{x},\bar{y},\bar{z})$. To furnish this, we apply Theorem 3.8 in [21] that provide conditions ensuring the SNC property of inverse images. By virtue of (4.6) and (4.7) one only needs to check that g is PSNC at $(\bar{x},\bar{y})$ if f is PSNC at this point. Indeed, taking sequences $(x_k^*,y_k^*)\in\hat{D}^*g(x_k,y_k)(u_k^*,v_k^*,z_k^*)$ with $(x_k^*,y_k^*)\stackrel{w^*}{\to}(0,0)$ and $\|(u_k^*,v_k^*,z_k^*)\|\to 0$, we get

$$(x_k^*,y_k^*)=(u_k^*,v_k^*)+(\hat{x}_k^*,\hat{y}_k^*) \text{ with } (\hat{x}_k^*,\hat{y}_k^*)\in\hat{D}^*f(x_k,y_k)(-z_k^*)$$

due to the representation

$$g(x,y)=(x,y,0)+(0,0,-f(x,y))$$

and the elementary equality rule for representing $\hat{D}^*g(x_k,y_k)$ in the above sum. This implies that $(\hat{x}_k^*,\hat{y}_k^*)\stackrel{w^*}{\to}(0,0)$ and hence $\|(\hat{x}_k^*,\hat{y}_k^*)\|\to 0$ by the PSNC property of f. Thus $\|(x_k^*,y_k^*)\|\to 0$ as well, i.e., g is PSNC at $(\bar{x},\bar{y})$.

If f is w^*-strictly Lipschitzian around $(\bar{x},\bar{y})$, the second qualification condition in the theorem is equivalent to (4.4) due to the scalarization formula (2.8). Then it is easy to show that (4.15) is equivalent to the simultaneous fulfillment of (4.4) and the first qualification condition of the theorem in this case. □

The results obtained above are expressed in general coderivative terms for mappings involved in describing variational systems. They can be particularly specified when the field Q in (1.1) is given in *subdifferential composite forms* like

$$Q=\partial(\varphi\circ g) \text{ and } Q=\partial\varphi\circ g,$$

which are typical in applications to parametric optimization, complementarity, variational and hemivariational inequalities, etc. In such cases resulting conditions involve *second-order subdifferentials* (or generalized Hessians)

$$\partial^2\varphi(\bar{x},\bar{y})(u):=(D^*\partial\varphi)(\bar{x},\bar{y})(u)$$

and can be expressed in terms of the initial data via second-order subdifferential calculus and computations available for special class of functions; see [4], [25], [15], and [16] for more details.

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